Student Number: ____________________ Teacher Name: ______________________

# PENRITH SELECTIVE HIGH SCHOOL

## MATHEMATICS DEPARTMENT

## HSC Trial Examination
## 2024
## Year 12 Mathematics Extension 2

**General Instructions:**

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided with this examination paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- No correction tape/white out allowed.

**Total marks: 100**

**Section I – 10 marks** (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II – 90 marks** (pages 6–11)

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

| | Multiple Choice | Q11 | Q12 | Q13 | Q14 | Q15 | Q16 | TOTAL |
|---|---|---|---|---|---|---|---|---|
| **Proof** | 3 /1 | a /2 | c /5 | b /4 | b /5 | c /5 | | /22 |
| **Complex Numbers** | 2, 5, 6 /3 | b /2 | b, d /7 | | a, c /10 | | | /22 |
| **Vectors** | 4, 7, 10 /3 | d /3 | | a /3 | | b /2 | | /11 |
| **Calculus** | 1 /1 | c /2 | a /3 | c /6 | | a /3 | b /7 | /22 |
| **Mechanics** | 8, 9 /2 | e /6 | | d /3 | | d /6 | a /6 | /23 |
| **TOTAL** | /10 | /15 | /15 | /16 | /15 | /16 | /13 | /100 |

## Section I

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which expression is equal to $\int_0^1 \frac{dx}{x^2+2x+2}$?

A. $\frac{\pi}{4}$

B. $-\frac{\pi}{4}$

C. $-\frac{\pi}{4}+\tan^{-1}2$

D. $\frac{\pi}{4}-\tan^{-1}2$

2 If $\text{Arg}\left(-\sqrt{3}+ai\right)=-\frac{5\pi}{6}$, find the value of the real number $a$.

A. $a=-1$

B. $a=1$

C. $a=0.5$

D. $a=-0.5$

3 The negation of the statement "$\forall m \in \mathbb{Z}, \exists n \geq m$ such that $n=m^2$ " is one of the following:

A. $\exists m \in \mathbb{Z}$, such that $n=m^2$, $\forall n \geq m$

B. $\forall m \in \mathbb{Z}$, such that $n \neq m^2$, $\forall n \geq m$

C. $\exists m \in \mathbb{Z}$, such that $n \neq m^2$, $\forall n \geq m$

D. $\forall m \in \mathbb{Z}$, such that $n \neq m^2$, $\exists n \geq m$

4 Consider a vector equation of a line in parametric form

$$\underset{\sim}{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + (1 - \sin(t))\mathbf{k}$$

Which one of the following statements is True?

A. The curve looks like $x^2 + y^2 = 1$ in the $xy$ −plane.

B. The curve looks like $z = 1 - y$ in the $yz$ −plane.

C. The curve looks like $x^2 + (z-1)^2 = 1$ in the $xz$ −plane.

D. ALL of the above.

5 If $P, Q, R$ represent the complex numbers $p = 1 + i, q = 2 + 6i, r = -1 + 7i$ respectively. Find the complex number $s$, represented by a point on an Argand diagram so that these four points form the vertices of parallelogram $PSQR$.

A. $s = 12i$

B. $s = -2 + 2i$

C. $s = 4$

D. Not enough information.

6 Consider the expression $S_n = 1 + z + z^2 \dots \dots .. + z^n$, where $z$ is any complex number of modulus 1. $S_n$ is also written as one of the following expressions:

A. $S_n = \frac{e^{in\theta} - 1}{e^{i\theta} - 1}$

B. $S_n = \frac{e^{i(n-1)\theta} - 1}{e^{i\theta} - 1}$

C. $S_n = \frac{e^{i(n-1)\theta} - 1}{e^{-i\theta} - 1}$

D. $S_n = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$

7 Consider a line represented by the vector equation $\underset{\sim}{r} = (\mathbf{i} - 2\mathbf{j}) + \mu(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and the sphere $|\underset{\sim}{v} - \underset{\sim}{a}| = 3, \underset{\sim}{a} = (2, -1, 3)$.

Which one of the following statements is true?

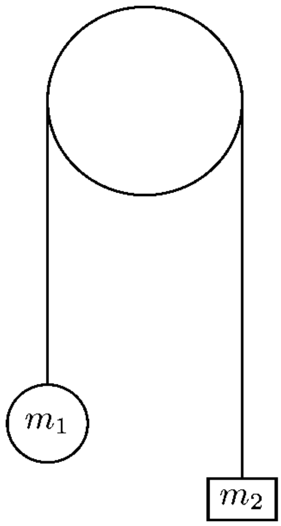
A. The line touches the sphere

B. The line intersects the sphere at two points

C. There is no point of intersection

D. Not enough information

8 The diagram below shows two objects with masses $m_1$ kg and $m_2$ kg, where $m_1 < m_2$, on either end of a light inextensible string that passes through a smooth pulley.




The acceleration of the heavier particle is given by one of the following expressions:

A. $a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$

B. $a = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) g$

C. $a = \left(\frac{m_2 - m_1}{m_2}\right) g$

D. $a = \left(\frac{m_2 + m_1}{m_1}\right) g$

9 A variable force of F newtons acts on a 3 kg mass so that it moves in a straight line. At time t seconds, $t \geq 0$, its velocity *v* metres per second and position *x* metres from the origin are given by $v = 3 - x^2$. It follows that

A. $F = -2x$

B. $F = -6x$

C. $F = 6x^3 - 18x$

D. $F = 9x - 3x^3$

10 A parametric vector equation is represented by

$$\underset{\sim}{r}(t) = \begin{pmatrix} t + \frac{1}{t} \\ t^2 + \frac{1}{t^2} \end{pmatrix}$$

Which of the following statement is correct?

A. The cartesian equation of the given vector equation is $y = x^2 - 2, t \leq 0$.

B. The cartesian equation of the given vector equation is $y = 2x^2 - 1, t \geq 0$

C. The cartesian equation of the given vector equation is $y = x^2 - 2, t > 0$.

D. The cartesian equation of the given vector equation is $y = 2x^2 - 1, t < 0$.

## End of Section I

## Section II

**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hours and 45 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use a separate Writing Booklet for Question 11.

(a) Let $n \in \mathbb{Z}$. Use the contrapositive to prove that if $n^3$ is even, then $n$ is even. **2**

(b) Find a complex number $z = x + iy$ such that $\operatorname{Im}(\bar{z}) + 2z = \frac{1}{1+i}$. **2**

(c) Evaluate $\int_0^{\frac{\pi}{4}} (2x + 1) \cos x \, dx$. **2**

(d) Let $\underset{\sim}{a} = 3\mathbf{i} + 2\mathbf{j} + m\mathbf{k}$ and $\underset{\sim}{b} = 4\mathbf{i} - \mathbf{j} + m^2\mathbf{k}$, where $m$ is a real constant. It is given that the scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\frac{74}{\sqrt{273}}$. **3**

Show that the value of $m$ is 4.

(e)
i. Prove that a particle moving according to the equation $|v| = \sqrt{-9x^2 + 12x + 32}$ is undergoing simple harmonic motion. **2**

ii. Find the period and amplitude of the motion. **2**

iii. Also, find the two positions of the particle when it is moving at half of its maximum speed. **2**

**End of Question 11**

**Question 12** (15 marks) Use a separate Writing Booklet for Question 12.

(a) Evaluate $\int_1^2 \frac{dx}{\sqrt{3+2x-x^2}}$. **3**

A line in the complex plane is given by $|z-1| = |z+2-3i|$, $z \in \mathbb{C}$.

(b) i. Show that the cartesian equation of the line is given by $y = x + 2$. **2**

ii. Sketch the line $|z-1| = |z+2-3i|$ and the circle $|z-1| = 3$ on the same Argand plane, labelling all the essential features including the points of intersection. **2**

(c) Assume that for any integers $a$, $b$ and $c$, if $a|c$, $b|c$ and $a$ and $b$ are co-prime, then $(ab)|c$.

**(DO NOT Prove)**

Given that $n$ is a positive integer.

i. Show that $n(n-1)(n+1)$ is always divisible by 6. **2**

ii. Show that $n(n-1)(n+1)$ is divisible by 30 unless $n$ is of the form $n = 5m \pm 2$ for some integer $m$. **3**

(d) It is given that $\sin\theta + \sin\varphi = 2\cos\left(\frac{\theta-\varphi}{2}\right)\sin\left(\frac{\theta+\varphi}{2}\right)$ (DO NOT Prove it)

i. Hence or otherwise, show that $\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta-\varphi}{2}\right)\cos\left(\frac{\theta+\varphi}{2}\right)$. **2**

ii. Hence, deduce that $e^{i\theta} + e^{i\varphi} = 2\cos\left(\frac{\theta-\varphi}{2}\right)e^{\frac{i}{2}(\theta+\varphi)}$. **1**

**End of Question 12**

**Question 13** (16 marks) Use a separate Writing Booklet for Question 13.

(a) Find the point of intersection between the two lines: **3**

$$\underset{\sim}{r} = (\mathbf{i} + 2\mathbf{k}) + \alpha(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\underset{\sim}{s} = (-2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \beta(-3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

where $\alpha$ and $\beta$ are any real numbers.

(b) i. Show that $\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$ **1**

ii. Use mathematical induction to prove that $\tan\left(\frac{(2n+1)\pi}{4}\right) = (-1)^n \ \forall n \in Z^+$ **3**

(c) Define for $n \geq 0$, $I_n = \int_0^\infty x^{n-1}e^{-x}dx$.

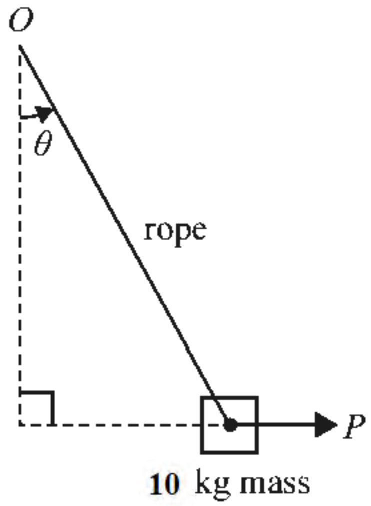
You may assume that for all real $n > 0$, $\lim_{x\to\infty} x^n e^{-x} = 0$.

i. Show that $I_{n+1} = nI_n$ **2**

ii. Deduce that $I_{n+1} = n!$ **2**

iii. Let $T_n = \int_0^1 (\ln x)^n dx$. Prove that $\frac{I_{n+1}}{T_n} = (-1)^n$. **2**

(d) A taut rope of length 1.25 m suspends a mass of 10 kg from a fixed point $O$. A horizontal force of P newtons displaces the mass by 1 m horizontally so that the taut rope is then at an angle of $\theta$ to the vertical.




Find the magnitude of the tension force in the rope in Newtons. ($g = 10m/s^2$) **3**

**End of Question 13**

**Question 14** (15 marks) Use a separate Writing Booklet for Question 14.

(a) Simplify the following expression, leaving the answer in Euler form. **2**

$$\frac{(\cos 3\theta + i\ \sin 3\theta)^2(\cos\theta - \ \mathrm{i}\sin\theta)^4}{(\cos 2\theta - i\ \sin 2\theta)^{-3}}$$

(b) On the curve $y = e^x, P$ and $Q$ are two points with $x$ −coordiantes $a$ and $b$ respectively, $a > b$.

i. Prove that $y = e^x$ is concave up for all $x$. **1**

ii. Explain why $\frac{1}{2}(e^a + e^b) > e^{\frac{1}{2}(a+b)}$. **2**

iii. Hence show that $e^a + e^b + e^c + e^d > 4e^{\frac{a+b+c+d}{4}}$ if $a > b > c > d$. **2**

(c) It is given that $\cos 6\theta = 32\ \cos^6\theta - 48\ \cos^4\theta + 18\ \cos^2\theta - 1$.

i. Find all the roots of the polynomial $32x^6 - 48x^4 + 18x^2 - 1 = 0$ **2**

ii. Show that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4}$ **2**

iii. Show that $\cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} = 1$ **2**

iv. Hence or otherwise, show that $\cos\frac{\pi}{12} + \cos\frac{5\pi}{12} = \frac{\sqrt{6}}{2}$. **2**

**End of Question 14**

**Question 15** (16 marks) Use a separate Writing Booklet for Question 15.

(a) Using the identity $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, **3**

evaluate $\int_0^{\pi} \frac{x \sin x \, dx}{1+\cos^2 x}$ .

(b) If $\underset{\sim}{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\underset{\sim}{b} = -m\mathbf{i} + \mathbf{j} + 2\boldsymbol{k}$, where $m$ is a real constant. Calculate the value(s) of $m$ so that the vector $\underset{\sim}{a} - \underset{\sim}{b}$ is perpendicular to $\underset{\sim}{b}$. **2**

(c)

i. Prove that $\sqrt{3}$ is an irrational number. **2**

ii. Consider the recurrence relation $T_1 = 2$, $T_n = \sqrt{1 + T_{n-1}}$ for $n \geq 2$. **3**
Using the mathematical induction, prove that $T_n$ is an irrational number for all $n \geq 2$.

(d) A car is initially at rest at a point $A$ which is 1 km to the right of Woolworths in Seven Hills Plaza. The car then starts moving in a straight line towards Woolworths.

For $x \neq 0$, the acceleration of the particle is given by $-\frac{k}{x^2}$, where $x$ is the distance (in kilometres) from the shop and $k$ is a positive constant.

i. Prove that $\frac{dx}{dt} = -\sqrt{\frac{2k(1-x)}{x}}$ . **2**

ii. Use the substitution $x = \cos^2\theta$, show that the time taken to reach a distance $D$ kilometres from Woolworths is given by **2**

$$t = \sqrt{\frac{2}{k}} \int_0^{\cos^{-1}\sqrt{D}} \cos^2\theta \; d\theta$$

iii. Show that $t = \sqrt{\frac{1}{2k}} \left(\sqrt{D - D^2} + \cos^{-1}\sqrt{D}\right)$. **2**

**End of Question 15**

**Question 16** (13 marks) Use a separate Writing Booklet for Question 16.

(a) The tide at a harbour can be modelled using simple harmonic motion. At the harbour, high tide is 12 metres and low tide is at 2 metres. It takes 4 hours to go from low tide to high tide. Initially at 2 am, it is at low tide. Let $t$ be measured in hours since 2 am.

i. Show that $x = 7 - 5\cos\left(\frac{t\pi}{4}\right)$. **3**

ii. A ship needs at least 5 metres of depth of water to safely enter the harbour. Find the earliest time that the ship may enter the harbour. **3**

(b) i. Find the real numbers $a, b$ and $c$ such that **2**

$$\frac{5-5x^2}{(1+2x)(1+x^2)} \equiv \frac{a}{1+2x} + \frac{bx+c}{1+x^2}$$

ii. Show that $\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx = \frac{1}{2}\left(\pi + \ln\left(\frac{27}{16}\right)\right)$. **3**

iii. Hence, evaluate $\int_0^{\frac{\pi}{4}} \frac{\cos 2x\,dx}{1+2\sin 2x+\cos 2x}$ using the substitution $t = \tan x$. **2**

**End of Question 16**

## END OF PAPER

## Multiple-choice answers

| | | | | |
|---|---|---|---|---|
| 1. C | 2. A | 3. C | 4. D | 5. C |
| 6. D | 7. B | 8. B | 9. C | 10. C |

**Question 11** (15 marks) Use a separate Writing Booklet for Question 11.

(a) Let $n \in \mathbb{Z}$. Use the contrapositive to prove that if $n^3$ is even, then $n$ is even. **2**

Contrapositive: If n is odd, then $n^3$ is odd. 1 mark

Let $n = 2k + 1, k \in \mathbb{Z}$

Then $n^3 = (2k+1)^3$

$= 8k^3 + 3(2k)^2 + 3(2k) + 1$

$= 2(4k^3 + 6k^2 + 3k) + 1$

$= 2P + 1$, where $P = 4k^3 + 6k^2 + 3, k \in \mathbb{Z}$ 1 mark

$\therefore$ **By contrapositive**, if $n^3$ is even, then $n$ is even.

Must state the last sentence or penalised.

(b) Find a complex number $z = x + iy$ such that $\text{Im}(\bar{z}) + 2z = \dfrac{1}{1+i}$. **2**

Let $z = x + iy$, where $x, y \in \mathbb{R}$.

Then $\bar{z} = x - iy$, $\text{Im}(\bar{z}) = -y$.

$\text{Im}(x - iy) \neq -iy$

$\text{Im}(x - iy) \neq y$

$$-y + 2(x + iy) = \frac{1}{1+i} \times \frac{1-i}{1-i}$$

$$-y + 2x + i2y = \frac{1-i}{2}$$ 1 mark

Equating the real and imaginary parts,

$2x - y = \dfrac{1}{2}$ [1]

$2y = -\dfrac{1}{2}$ [2]

$2 \times [1] + [2]$, $4x = \dfrac{1}{2}$

$x = \dfrac{1}{8}$

From [1], $\dfrac{1}{4} - y = \dfrac{1}{2}$

$y = -\dfrac{1}{4}$

$\therefore z = \dfrac{1}{8} - \dfrac{1}{4}i$ 1 mark

**Question 11** (Continues)

(c) Evaluate $\int_0^{\frac{\pi}{4}}(2x+1)\cos x\,dx$. **2**

$u = 2x+1, \qquad v' = \cos x$
$u' = 2, \qquad v = \sin x$

$$\int_0^{\frac{\pi}{4}}(2x+1)\cos x\,dx = [(2x+1)\sin x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2\sin x\,dx \qquad \text{(by parts)}$$ 1 mark

$$= \left(\left(\frac{\pi}{2}+1\right)\sin\frac{\pi}{4} - \sin 0\right) + 2[\cos x]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{2}+1\right)\frac{1}{\sqrt{2}} + 2\left[\cos\frac{\pi}{4} - \cos 0\right]$$

Careless error: $\cos 0 \neq 0$

$$= \frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} - 2$$

$$= \frac{\pi+6}{2\sqrt{2}} - 2$$ 1 mark

(d) Let $\underset{\sim}{a} = 3\mathbf{i} + 2\mathbf{j} + m\mathbf{k}$ and $\underset{\sim}{b} = 4\mathbf{i} - \mathbf{j} + m^2\mathbf{k}$, where $m$ is a real constant. It is given that the scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\frac{74}{\sqrt{273}}$. **3**

Show that the value of $m$ is 4.

$$\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|}\hat{\underset{\sim}{b}}$$

$$\text{scalar projection} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$\frac{74}{\sqrt{273}} = \frac{3\times 4 + 2\times(-1) + m\times m^2}{\sqrt{4^2+(-1)^2+(m^2)^2}}$$ 1 mark

$$\frac{74}{\sqrt{273}} = \frac{10+m^3}{\sqrt{17+m^4}}$$

Equating the **numerators** and **denominators**,

$74 = 10 + m^3, \qquad 273 = 17 + m^4$
$m^3 = 64 \qquad 256 = m^4$
$m = 4 \qquad m = 4$
$\therefore m = 4$

2 marks

**Question 11** (Continues)

(e) (i) Prove that a particle moving according to the equation $|v| = \sqrt{-9x^2 + 12x + 32}$ is undergoing simple harmonic motion. **2**

$|v| = \sqrt{-9x^2 + 12x + 32}$

Squaring both sides,

$v^2 = -9x^2 + 12x + 32$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= \frac{d}{dx}\left(-\frac{9}{2}x^2 + 6x + 16\right)$$

1 mark

$$= -9x + 6$$
$$\ddot{x} = -9(x - \frac{2}{3})$$

$\therefore$ Since $\ddot{x} \propto x$, the motion is undergoing simple harmonic motion. 1 mark

Those who used $v^2 = n^2(a^2 - (x-c)^2)$ for (i) and (ii), must derive it. Or penalised.

$$v^2 = -9x^2 + 12x + 32$$
$$= -9\left(x^2 - \frac{4}{3}x - \frac{32}{9}\right)$$
$$= -9\left(x^2 - \frac{4}{3}x + \frac{4}{9} - \frac{4}{9} - \frac{32}{9}\right)$$
$$= 9\left(4 - \left(x - \frac{2}{3}\right)^2\right)$$

(ii) Find the period and amplitude of the motion. **2**

$n^2 = 9,$

$n = 3$ (since $n > 0$)

$$T = \frac{2\pi}{n}$$
$$\therefore T = \frac{2\pi}{3}$$

1 mark

centre $x = \frac{2}{3}$,

substitute $v = 0$ for the range,

$-9x^2 + 12x + 32 = 0$

$$x = \frac{-12 \pm \sqrt{12^2 - 4(-9)(32)}}{2(-9)}$$
$$x = \frac{-12 \pm 36}{-18}$$
$$x = -\frac{4}{3}, \frac{8}{3}$$
$$\therefore \text{amplitude} = \frac{8}{3} - \frac{2}{3}$$
$$= 2$$

1 mark

(iii) Also, find the two positions of the particle when it is moving at half of its maximum speed. **2**

Speed is maximum at the centre of the motion $x = \frac{2}{3}$

$$|v| = \sqrt{-9\left(\frac{2}{3}\right)^2 + 12\left(\frac{2}{3}\right) + 32}$$
$$= 6$$

$\therefore$ Maximum speed $= 6$

Substitute $v = 3$,

$3^2 = -9x^2 + 12x + 32$ 1 mark

$9x^2 - 12x - 23 = 0$

$$x = \frac{12 \pm \sqrt{12^2 - 4(9)(-23)}}{2(9)}$$
$$= \frac{12 \pm \sqrt{972}}{18}$$
$$= \frac{12 \pm 18\sqrt{3}}{18}$$
$$\therefore x = \frac{2 + 3\sqrt{3}}{3} \text{ and } \frac{2 - 3\sqrt{3}}{3}$$

1 mark

**End of Question 11**

**Question 12** (15 marks) Use a separate Writing Booklet for Question 12.

(a) Evaluate $\displaystyle\int_1^2 \frac{dx}{\sqrt{3+2x-x^2}}$ **3**

$$\int_1^2 \frac{dx}{\sqrt{3+2x-x^2}} = \int_1^2 \frac{1}{\sqrt{4-(1-2x+x^2)}}dx$$ ⟵ Show this step.

$$= \int_1^2 \frac{1}{\sqrt{2^2-(x-1)^2}}dx$$ 1 mark

$$= \left[\sin^{-1}\frac{x-1}{2}\right]_1^2$$ 1 mark

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$$ ⟵ Must show the substitution.

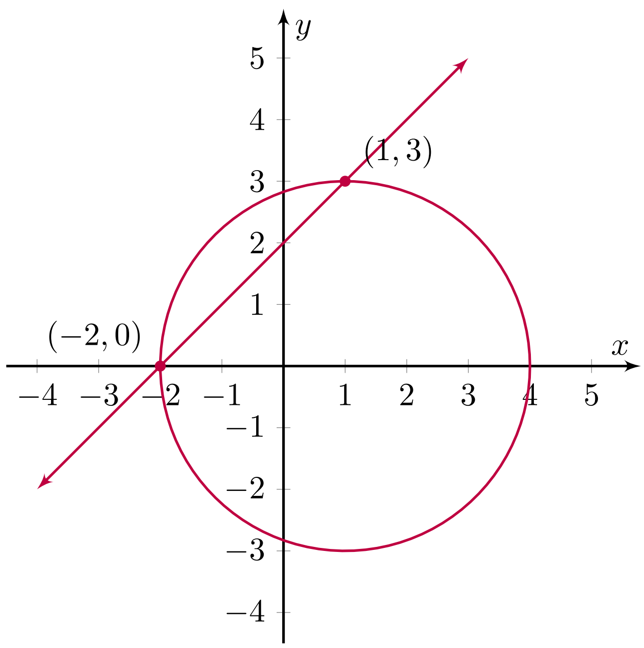
$$= \frac{\pi}{6}$$ 1 mark

(b) A line in the complex plane is given by $|z-1| = |z+2-3i|, z \in \mathbb{C}$.
(i) Show that the cartesian equation of the line is given by $y = x+2$. **2**

Let $z = x+iy$
$|x+iy-1| = |x+iy+2-3i|$
$|(x-1)+iy| = |(x+2)+i(y-3)|$
$(x-1)^2+y^2 = (x+2)^2+(y-3)^2$ 1 mark
$x^2-2x+1+y^2 = x^2+4x+4+y^2-6y+9$
$-2x+1 = 4x-6y+13$
$6y = 6x+12$
$\therefore y = x+2$
1 mark

<u>Alternative:</u>
Find the perpendicular bisector of the segment joining $(-2, 3)$ and $(1, 0)$.

(ii) Sketch the line $|z-1| = |z+2-3i|$ and the circle $|z-1| = 3$ on the same Argand plane, labelling all the essential features including the points of intersection. **2**



**Question 12** (Continues)

(c) Assume that for any integers $a, b$ and $c$, if $a|c$, $b|c$ and $a$ and $b$ are co-prime, then $(ab)|c$.
**(DO NOT Prove)**

Given that $n$ is a positive integer.

(i) Show that $n(n-1)(n+1)$ is always divisible by 6. **2**

Poor explanation.

$n(n-1)(n+1)$ is divisible by 6 if it is divisible by 2 and 3.

For any three consecutive numbers, starting with an odd number or an even number, there is always at least one even number and one multiple of 3. 1 mark

Thus $n(n-1)(n+1)$ is divisible by 2 and 3.

$\therefore$ $n(n-1)(n+1)$ is also divisible by 6 since $\text{LCM}(2, 3) = 6$. 1 mark

Alternative

Consider:

$${}^{n+1}C_3 = \frac{(n+1)!}{3!\,(n-2)!}$$

$${}^{n+1}C_3 = \frac{(n+1)n(n-1)(n-2)!}{6(n-2)!}$$

$${}^{n+1}C_3 = \frac{(n+1)n(n-1)}{6}$$

Since ${}^{n+1}C_3$ is an integer, $(n+1)n(n-1)$ is divisible by 6.

(ii) Show that $n(n-1)(n+1)$ is divisible by 30 unless $n$ is of the form $n = 5m \pm 2$ for some integer $m$. **3**

$n(n-1)(n+1)$ is divisible by 30 if it is divisible by 5 and 6. Must state this.

Cannot be proven by contrapositive. The statement is not an implication. You are to show whether $\boldsymbol{n(n-1)(n+1)}$ is divisible by 30 or not.

When $\boldsymbol{n(n-1)(n+1)}$ is not divisible by 30 when $\boldsymbol{n = 5m \pm 2}$, there is no guarantee that it will be divisible by all other positive integer $\boldsymbol{n}$.

$n(n-1)(n+1)$ is divisible by 6. (from (i))

Now show that if it is divisible by 5:

If $\boldsymbol{n = 5m \pm 1}$, $n \pm 1 = 5m$
Then $n(n-1)(n+1) = n(5m)(5m)$
$= 5(5m^2 n)$ which is divisible by 5.
$\therefore$ $n(n-1)(n+1)$ is divisible by 5 if $n = 5m \pm 1$. [1]

If $\boldsymbol{n = 5m \pm 4}$,
$= 5m + 5 - 1$ or $5m - 5 + 1$
$= 5(m+1) - 1$ or $5(m-1) + 1$
$= 5P - 1$ or $5Q + 1$, where $P, Q \in \mathbb{Z}$
$\therefore$ $n(n-1)(n+1)$ is divisible by 5 if $n = 5m \pm 4$. (from [1])

If $\boldsymbol{n = 5m + 2}$,
$n(n-1)(n+1) = (5m+2)(5m+1)(5m+3)$ which is not divisible by 5.

If $\boldsymbol{n = 5m - 2}$,
$n(n-1)(n+1) = (5m-2)(5m-3)(5m-1)$ which is not divisible by 5.

If $\boldsymbol{n = 5m \pm 3}$,
$= 5m + 5 - 2$ or $5m - 5 + 2$
$= 5(m+1) - 2$ or $5(m-1) + 2$
$= 5P - 2$ or $5Q + 2$, where $P, Q \in \mathbb{Z}$
$\therefore$ $n(n-1)(n+1)$ is **not** divisible by 5 if $n = 5m \pm 3$ which is **equivalent to $\boldsymbol{5m \pm 2}$**.

$\therefore$ $n(n-1)(n+1)$ is divisible by 5 unless $n = 5m \pm 2, m \in \mathbb{Z}$.

Thus $n(n-1)(n+1)$ is divisible by 30 unless $n = 5m \pm 2, m \in \mathbb{Z}$.

Common error:

Shown that $\boldsymbol{n(n-1)(n+1)}$ is not divisible by 30 when $\boldsymbol{n = 5m \pm 2}$.

However, did not show for $\boldsymbol{n = 5m \pm 1}$, $\boldsymbol{5m \pm 3}$ and $\boldsymbol{5m \pm 4}$.

**Question 12** (Continues)

(d) It is given that $\sin\theta + \sin\varphi = 2\cos\left(\frac{\theta-\varphi}{2}\right)\sin\left(\frac{\theta+\varphi}{2}\right)$ (DO NOT Prove it)

(i) Hence or otherwise, show that $\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta-\varphi}{2}\right)\cos\left(\frac{\theta+\varphi}{2}\right)$. **2**

$\sin\theta + \sin\varphi = 2\cos\left(\frac{\theta-\varphi}{2}\right)\sin\left(\frac{\theta+\varphi}{2}\right)$

$\mathbf{\cos(90-\theta)} + \mathbf{\cos(90-\varphi)} = 2\cos\left(\frac{\theta-\varphi}{2}\right)\mathbf{\cos\left(90-\frac{\theta+\varphi}{2}\right)}$ [1] (Complementary angle ratio)

Let $A = 90-\theta$ and $B = 90-\varphi$

$\theta = 90 - A$ $\quad\varphi = 90 - B$

$\theta - \varphi = B - A$ $\quad\theta + \varphi = 180 - (A+B)$

From [1], $\cos A + \cos B = 2\cos\left(\frac{B-A}{2}\right)\cos\left(\frac{B+A}{2}\right)$

$\therefore \cos\theta + \cos\varphi = 2\cos\left(\frac{\theta-\varphi}{2}\right)\cos\left(\frac{\theta+\varphi}{2}\right)$

Alternative 1:

$$\begin{aligned}
\text{LHS} &= \cos\theta + \cos\varphi \\
&= \sin(90-\theta) + \sin(90-\varphi) && \text{(complementary angle ratio)} \\
&= 2\cos\left(\frac{(90-\theta)-(90-\varphi)}{2}\right)\sin\left(\frac{(90-\theta)+(90-\varphi)}{2}\right) && \text{(given)} \\
&= 2\cos\left(\frac{\varphi-\theta}{2}\right)\sin\left(\frac{180-\theta-\varphi}{2}\right) \\
&= 2\cos\left(-\frac{\theta-\varphi}{2}\right)\sin\left(90-\frac{\theta+\varphi}{2}\right) \\
&= 2\cos\left(\frac{\theta-\varphi}{2}\right)\cos\left(\frac{\theta+\varphi}{2}\right) && \text{(since } \cos\theta \text{ is even \& complementary angle ratio)} \\
&= \text{RHS}
\end{aligned}$$

Alternative 2:

$$\begin{aligned}
\text{RHS} &= 2\cos\left(\frac{\theta-\varphi}{2}\right)\cos\left(\frac{\theta+\varphi}{2}\right) \\
&= 2\left(\cos\frac{\theta}{2}\cos\frac{\varphi}{2} + \sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right)\left(\cos\frac{\theta}{2}\cos\frac{\varphi}{2} - \sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right) && \text{(supplementary angle ratio)} \\
&= 2\left(\cos^2\frac{\theta}{2}\cos^2\frac{\varphi}{2} + \sin^2\frac{\theta}{2}\sin^2\frac{\varphi}{2}\right) && \text{(difference of two squares)} \\
&= 2\left(\frac{1}{2}(1+\cos\theta)\times\frac{1}{2}(1+\cos\varphi) - \frac{1}{2}(1-\cos\theta)\times\frac{1}{2}(1-\cos\varphi)\right) && \text{(double angle rule)} \\
&= \frac{1}{2}\left((1+\cos\theta)(1+\cos\varphi) - (1-\cos\theta)(1-\cos\varphi)\right) \\
&= \frac{1}{2}(1+\cos\varphi+\cos\theta+\cos\theta\cos\varphi - 1 + \cos\varphi + \cos\theta - \cos\theta\cos\varphi) \\
&= 2(2\cos\theta + 2\cos\varphi) \\
&= \cos\theta + \cos\varphi \\
&= \text{LHS}
\end{aligned}$$

**Question 12** (Continues)

(ii) Hence, deduce that $e^{i\theta} + e^{i\varphi} = 2\cos\left(\frac{\theta - \varphi}{2}\right)e^{\frac{i}{2}(\theta+\varphi)}$. **1**

$$\begin{aligned}
\text{LHS} &= e^{i\theta} + e^{i\varphi} \\
&= (\cos\theta + i\sin\theta) + (\cos\varphi + i\sin\varphi) \\
&= (\cos\theta + \cos\varphi) + i(\sin\theta + \sin\varphi) \\
&= 2\cos\left(\frac{\theta-\varphi}{2}\right)\cos\left(\frac{\theta+\varphi}{2}\right) + 2i\cos\left(\frac{\theta-\varphi}{2}\right)\sin\left(\frac{\theta+\varphi}{2}\right) \qquad \text{(from (i))} \\
&= 2\cos\left(\frac{\theta-\varphi}{2}\right)\left(\cos\left(\frac{\theta+\varphi}{2}\right) + i\sin\left(\frac{\theta+\varphi}{2}\right)\right) \\
&= 2\cos\left(\frac{\theta-\varphi}{2}\right)e^{i\left(\frac{\theta+\varphi}{2}\right)} \\
&= 2\cos\left(\frac{\theta-\varphi}{2}\right)e^{\frac{i}{2}(\theta+\varphi)} \\
&= \text{RHS}
\end{aligned}$$

**End of Question 12**

**Question 13** (16 marks) Use a separate Writing Booklet for Question 13.

(a) Find the point of intersection between the two lines: **3**

$$\underset{\sim}{r} = (\mathbf{i} + 2\mathbf{k}) + \alpha(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$
$$\underset{\sim}{s} = (-2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \beta(-3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

where $\alpha$ and $\beta$ are any real numbers.

For the point of intersection, $\underset{\sim}{r} = \underset{\sim}{s}$.

$$\begin{pmatrix} 1+2\alpha \\ 0-\alpha \\ 2+\alpha \end{pmatrix} = \begin{pmatrix} -2-3\beta \\ 2+2\beta \\ 1-\beta \end{pmatrix}$$ 1 mark

$1 + 2\alpha = -2 - 3\beta$ [1]
$-\alpha = 2 + 2\beta$ [2]
$2 + \alpha = 1 - \beta$ [3]

Label the equations to show steps.

[2] + [3], $2 = 3 + \beta$ [1]
$\beta = -1$
From [2], $-\alpha = 2 + 2(-1)$
$\alpha = 0$

1 mark

Substitute $\alpha = 0$ & $\beta = -1$ into [1] to verify,
LHS $= 1 + 2(0)$, RHS $= -2 - 3(-1)$
$= 1$ $= 1$
$=$ LHS

When $\alpha = 0$, $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\therefore$ The point of intersection is $(1, 0, 2)$.

1 mark

Must check the value of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with the 3rd equation.

(b) (i) Show that $\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$. **1**

Done well.

$$\tan\left(\frac{\pi}{2}+\theta\right) = \frac{\sin\left(\frac{\pi}{2}+\theta\right)}{\cos\left(\frac{\pi}{2}+\theta\right)}$$
$$= \frac{\sin\frac{\pi}{2}\cos\theta + \cos\frac{\pi}{2}\sin\theta}{\cos\frac{\pi}{2}\cos\theta - \sin\frac{\pi}{2}\sin\theta}$$
$$= \frac{\cos\theta}{-\sin\theta}$$
$$= -\cot\theta$$

<u>Alternative</u>:

LHS $= \tan\left(\frac{\pi}{2}+\theta\right)$
$= \tan\left(\frac{\pi}{2} - (-\theta)\right)$
$= \cot(-\theta)$ (complementary angle ratio)
$=$ RHS

**Question 13** (Continues)

(b)

(ii) Use mathematical induction to prove that $\tan\left(\frac{(2n+1)\pi}{4}\right) = (-1)^n \ \forall n \in Z^+$. **3**

Step 1 Prove true for $n = 1$.

$$\text{LHS} = \tan\frac{3\pi}{4}, \qquad \text{RHS} = (-1)^1$$
$$= -\tan\frac{\pi}{4} \qquad\qquad = -1$$
$$= -1 \qquad\qquad = \text{LHS}$$

$\therefore$ The statement is true for $n = 1$. 1 mark

Step 2 Assume true for $n = k$, where $k \in Z^+$.

i.e. $\tan\left(\frac{(2k+1)\pi}{4}\right) = (-1)^k$

Now prove true for $n = k + 1$.

i.e. $\tan\left(\frac{(2k+3)\pi}{4}\right) = (-1)^{k+1}$

$$\begin{aligned}
\text{LHS} &= \tan\left(\frac{(2k+3)\pi}{4}\right) \\
&= \tan\left(\frac{(2k+1)\pi + 2\pi}{4}\right) \\
&= \tan\left(\frac{(2k+1)\pi}{4} + \frac{\pi}{2}\right) \\
&= -\cot\left(\frac{(2k+1)\pi}{4}\right) && \left(\text{since } \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta \text{ from (i)}\right) \\
&= -\frac{1}{\tan\left(\frac{(2k+1)\pi}{4}\right)} \\
&= -\frac{1}{(-1)^k} && \text{(by the assumption)} \\
&= -(-1)^k \\
&= (-1)^{k+1} \\
&= \text{RHS}
\end{aligned}$$

1 mark

Explain the key steps.

1 mark

Step 3 By the principle of mathematical induction, the statement is true for all integers $n \geq 1$.

**Question 13** (Continues)

(c) Define for $n \geq 0, I_n = \int_0^{\infty} x^{n-1} e^{-x}\, dx$.

You may assume that for all real $n > 0$, $\lim\limits_{x \to \infty} x^n e^{-x} = 0$.

(i) Show that $I_{n+1} = nI_n$ **2**

$$I_{n+1} = \int_0^{\infty} x^n e^{-x}\, dx$$

$u = x^n$, $\quad v' = e^{-x}$

$u' = nx^{n-1}$, $\quad v = -e^{-x}$

$$\begin{aligned} \text{LHS} &= I_{n+1} \\ &= [-x^n e^{-x}]_0^{\infty} + \int_0^{\infty} nx^{n-1} e^{-x}\, dx \qquad \text{(by parts)} \end{aligned}$$

1 mark

$$\begin{aligned} &= (-\infty^n e^{-\infty} + 0^n e^0) + n\int_0^{\infty} x^{n-1} e^{-x}\, dx \\ &= 0 + nI_n \qquad \left(\text{since } \lim_{x \to \infty} x^n e^{-x} = 0\right) \end{aligned}$$

1 mark

$$\begin{aligned} &= nI_n \\ &= \text{RHS} \end{aligned}$$

(ii) Deduce that $I_{n+1} = n!$ **2**

$$\begin{aligned} \text{LHS} &= I_{n+1} \\ &= nI_n \qquad \text{(from (i))} \\ &= n(n-1)I_{n-1} \\ &= n(n-1)(n-2)I_{n-2} \\ &= n(n-1)(n-2) \ldots 1I_1 \end{aligned}$$

1 mark

$$\begin{aligned} I_1 &= \int_0^{\infty} x^0 e^{-x}\, dx \\ &= [-e^{-x}]_0^{\infty} \\ &= (-e^{-\infty} + e^0) \\ &= (0+1) \\ &= 1 \end{aligned}$$

Must find the value of $I_1$

1 mark

$$\begin{aligned} I_{n+1} &= n(n-1)(n-2) \ldots 1 \\ &= n! \\ &= \text{RHS} \end{aligned}$$

**Question 13** (Continues)

(c) (iii) Let $T_n = \int_0^1 (\ln x)^n \, dx$. Prove that $\frac{I_{n+1}}{T_n} = (-1)^n$. **2**

$T_n = \int_0^1 (\ln x)^n \, dx$

$u = (\ln x)^n, \qquad v' = 1$

$u' = n(\ln x)^{n-1}\frac{1}{x} \qquad v = x$

$T_n = [x(\ln x)^n]_0^1 - \int_0^1 n(\ln x)^{n-1} \, dx$ [1] (by parts)

$= 0 - nT_{n-1}$
$= -nT_{n-1}$

1 mark

$T_n = -n \times -(n-1)T_{n-2}$
$= -n \times -(n-1) \times -(n-2)T_{n-3}$
$= -n \times -(n-1) \times -(n-2) \times -(n-3) \times \ldots \times -2T_1$

$T_1 = [x \ln x]_0^1 - \int_0^1 1 \, dx$ (from [1])
$= (0-0) - [x]_0^1$
$= -1 - 0$
$= -1$

$T_n = -n \times -(n-1) \times -(n-2) \times -(n-3) \times \ldots \times -2 \times -1$
$\therefore T_n = (-1)^n n!$

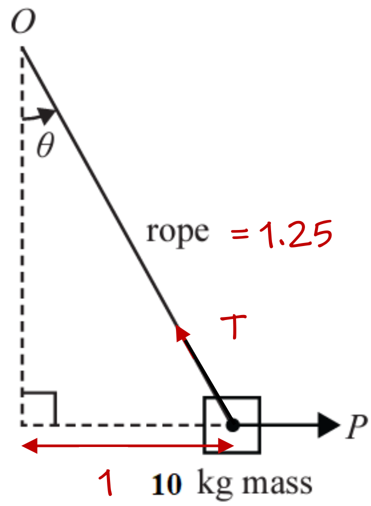
$\therefore \frac{I_{n+1}}{T_n} = \frac{n!}{(-1)^n n!}$ (since $I_{n+1} = n!$ from (ii))
$= \frac{1}{(-1)^n}$
$= (-1)^n$

1 mark

Must find the value of $T_1$

**Question 13** (Continues)

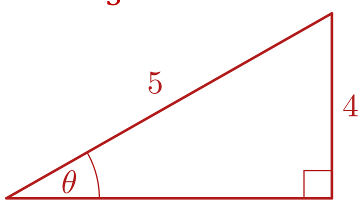
(d) A taut rope of length 1.25 m suspends a mass of 10 kg from a fixed point $O$. A horizontal force of P newtons displaces the mass by 1 m horizontally so that the taut rope is then at an angle of $\theta$ to the vertical.




Find the magnitude of the tension force in the rope in Newtons. ($g = 10 \text{ m}/s^2$) **3**

$\sin\theta = \dfrac{1}{1.25}$

$\sin\theta = \dfrac{4}{5}$




$\sqrt{5^2 - 4^2} = 3$

$\therefore \cos\theta = \dfrac{3}{5}$

1 mark

Resolving the forces vertically,

$T\cos\theta = 10g$

1 mark

$T = \dfrac{100}{\cos\theta}$

$= \dfrac{100}{\frac{3}{5}}$

$= \dfrac{500}{3}\text{ N} \qquad \left(\text{or } 166\dfrac{2}{3}\text{ N}\right)$

1 mark

**End of Question 13**

**Question 14** (15 marks) Use a separate Writing Booklet for Question 14.

(a) Simplify the following expression, leaving the answer in Euler form. **2**

$$\frac{(\cos 3\theta + i\ \sin 3\theta)^2(\cos\theta -\ \mathrm{i}\sin\theta)^4}{(\cos 2\theta - i\ \sin 2\theta)^{-3}}$$

Using D'Moivre theorem $(\cos\theta + i\sin\theta\ )^n = \cos n\theta + i\sin n\theta$ for all positive integers n.

In euler form, $\left(e^{i\theta}\right)^n = e^{in\theta}$

Therefore,

$\frac{(\cos 3\theta + i\ \sin 3\theta)^2(\cos\theta - \mathrm{i}\sin\theta)^4}{(\cos 2\theta - i\ \sin 2\theta)^{-3}}$

$= \frac{e^{i3\theta^2} \times e^{-i\theta^4}}{e^{-2i\theta^{-3}}}$ (Make sure students mention why $e^{-i\theta^4} = (\cos\theta -\ \mathrm{i}\sin\theta)^4$

$= e^{-4i\theta}$

Comments:

Many students lost marks for not explaining why $(\cos\theta -\ \mathrm{i}\sin\theta) = e^{-i\theta}$

A number of errors were noticed while simplifying surds

Very few students left the correct answer in mod-cis form, hence lost a mark

Overall, it was a poor attempt

Learning Strategies:

Students need to

- revise simplifying surds
- apply D'Moivre theorem correctly
- apply odd/even properties of a function.
- Explain how $(\cos\theta -\ \mathrm{i}\sin\theta) = e^{-i\theta}$.

(b) On the curve $y = e^x$, $P$ and $Q$ are two points with $x$ −coordiantes $a$ and $b$ respectively, $a > b$.

i. Prove that $y = e^x$ is concave up for all $x$. **1**

For $y = e^x$, $y' = e^x$ and $y'' = e^x$

As $e^x$ is always positive for all values of $x, y'' > 0$.

Therefore, the graph $y = e^x$ is concave up.

Common mistakes:

- Some students did not find second derivative. **2**
- Few students wrote in words rather than proving mathematically.

ii. Explain why $\frac{1}{2}(e^a + e^b) > e^{\frac{1}{2}(a+b)}$.

$\left(e^{\frac{a}{2}} - e^{\frac{b}{2}}\right)^2 > 0$ as $e^x$ is always positive. For $a > b, e^a > e^b$ as it is concave up (part (i))

Therefore, $e^a - 2\, e^{\frac{a}{2}} e^{\frac{b}{2}} + e^b > 0$

Rearranging, $e^a + e^b > 2\, e^{\frac{a}{2}} e^{\frac{b}{2}}$ **2**

$\frac{1}{2}\ (e^a + e^b) > e^{\frac{a}{2}} e^{\frac{b}{2}}$ , hence proved.

Teachers Comments:

- Students were using AM/GM property. Teachers have specifically mentioned while teaching the topic to prove it, if you wish to apply.
- Many students messed up the working.

iii. Hence show that $e^a + e^b + e^c + e^d > 4e^{\frac{a+b+c+d}{4}}$ if $a > b > c > d$.

$e^a + e^b > 2\, e^{\frac{a}{2}} e^{\frac{b}{2}}$ using part (ii)

Similarly, $e^c + e^d > 2\, e^{\frac{c}{2}} e^{\frac{d}{2}}$ for $a > b > c > d$, increasing function.

Adding the two equations together,

$e^a + e^b + e^c + e^d > 2(e^{\frac{a}{2}}\, e^{\frac{b}{2}} + e^{\frac{c}{2}}\, e^{\frac{d}{2}})$

$$= 2(e^{\frac{1}{2}(a+b)} + e^{\frac{1}{2}(c+d)})$$

$>2(2e^{\frac{1}{2}\frac{(a+b+c+d)}{2}})$ using the repeated application of part (ii)

where new $a = \frac{1}{2}(a+b)$ and new b$= \frac{1}{2}(c+d)$

therefore, $e^a + e^b + e^c + e^d > 4e^{\frac{(a+b+c+d)}{4}})$ . Hence, proved.

Teachers Comments:

- Some students got the previous part incorrect but used the correct expression to prove/show this part. Well done!!
- Overall okay.

(c) It is given that $\cos 6\theta = 32\ \cos^6\theta - 48\ \cos^4\theta + 18\ \cos^2\theta - 1$.

i. Find all the roots of the polynomial $32x^6 - 48x^4 + 18x^2 - 1 = 0$ **2**

Let $x = \cos\theta$, therefore $\cos 6\theta = \ 32x^6 - 48x^4 + 18x^2 - 1$. In order to solve the polynomial, it is enough to solve for $\cos 6\theta = 0$.

$6\theta = 0,$ this means $6\theta = \frac{(2k+1)\pi}{2}$ for $k \in \mathbb{Z}$.

$\theta = \frac{(2k+1)\pi}{12}$ for $k \in \mathbb{Z}$

For $k = 0, \theta = \frac{\pi}{12}$  $k = 1, \theta = \frac{3\pi}{12}$  $k = -1, \theta = -\frac{\pi}{12}$

$k = 2, \theta = \frac{5\pi}{12}$  $k = -2, \theta = -\frac{3\pi}{12}$  $k = 3, \theta = \frac{7\pi}{12}$ which is same as $-\frac{5\pi}{12}$.

Therefore, the roots of the polynomial are $\cos\pm\frac{\pi}{12}, \cos\pm\frac{3\pi}{12}, \cos\pm\frac{5\pi}{12}$.

Teachers Comments:

- Some students found the roots by letting $k = 1, 2, \ldots\ldots, 6$
- Overall okay attempt.

ii. Show that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4}$

Using the product of roots:

$$\cos\frac{\pi}{12}\times\cos-\frac{\pi}{12}\times\cos\frac{3\pi}{12}\times\cos-\frac{3\pi}{12}\times\cos\frac{5\pi}{12}\times\cos-\frac{5\pi}{12}=\frac{1}{32}$$ 2

As $\cos(-\theta) = \cos\theta$ (even function),

$\cos^2\frac{\pi}{12}\times\cos^2\frac{3\pi}{12}\times\cos^2\frac{5\pi}{12}=\frac{1}{32}$

$\cos\frac{3\pi}{12} = 1/\sqrt{2}$

$\therefore \cos^2\frac{\pi}{12}\times\frac{1}{2}\times\cos^2\frac{5\pi}{12}=\frac{1}{32}$

$\therefore \cos^2\frac{\pi}{12}\times\cos^2\frac{5\pi}{12}=\frac{1}{16}$ taking square root on both sides, considering positive value as these angles are acute angles. We get $\cos\frac{\pi}{12}\cos\frac{5\pi}{12}=\frac{1}{4}$

## Method II:

- Some students used sums to products method to get to the answer.

$$\cos\frac{\pi}{12}\times\cos\frac{5\pi}{12}=\frac{1}{2}\left(\cos\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)+\cos\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)\right)$$ 2

Which also provides the answer.

- Many students got the answer, but the working was messy.
- Overall okay

iii. Show that $\cos^2\frac{\pi}{12}+\cos^2\frac{5\pi}{12}=1$

Method 1: take the transformation $y = x^2$,

the new polynomial will be $P(y) = 32y^3 - 48y^2 + 18y - 1 = 0$

the roots of the new polynomial are $\cos^2\frac{\pi}{12}, \cos^2\frac{3\pi}{12}, \cos^2\frac{5\pi}{12}$

sum of the roots: $\cos^2\frac{\pi}{12}+\cos^2\frac{3\pi}{12}+\cos^2\frac{5\pi}{12}=\frac{48}{32}$ 2

$\cos^2\frac{\pi}{12}+\cos^2\frac{5\pi}{12}=\frac{48}{32}-\frac{1}{2}$

Which gives, $\cos^2\frac{\pi}{12}+\cos^2\frac{5\pi}{12}=1$

Teachers Comment:

- Majority of the students did by taking products of the roots taking two at a time.
- Overall, it was a good attempt.

iv. Hence or otherwise, show that $\cos\frac{\pi}{12} + \cos\frac{5\pi}{12} = \frac{\sqrt{6}}{2}$.

Using the perfect square method technique,

$$\left(\left(\cos\frac{5\pi}{12} + \cos\frac{\pi}{12}\right)\right)^2 = \cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} + 2\cos\frac{\pi}{12} \times \cos\frac{5\pi}{12}$$

$$= 1 + 2\left(\frac{1}{4}\right)$$

Therefore, $\left(\cos\frac{5\pi}{12} + \cos\frac{\pi}{12}\right) = \sqrt{\frac{3}{2}}$,

As both the angles are acute, their angle sum is positive, therefore, we choose the positive square root.

Teachers Comment:

- Few students used sums to products and got the answer
- Few students lost one mark for not mentioning the reason for choosing the positive over the negative square root.
- Overall, it was well done.

**End of Question 14**

**Question 15** (16 marks) Use a separate Writing Booklet for Question 15.

(a) Using the identity $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, **3**

evaluate $\int_o^{\pi} \frac{x\ \sin x\ dx}{1+\cos^2 x}$.

$\int_o^{\pi} \frac{x\ \sin x\ dx}{1+\cos^2 x} = \int_o^{\pi} \frac{(\pi-x)\ \sin x\ dx}{1+\cos^2(\pi-x)}$ using the given identity

$= \int_o^{\pi} \frac{\pi\ \sin x\ dx}{1+\cos^2 x} - \int_o^{\pi} \frac{x\ \sin x\ dx}{1+\cos^2 x}$

$= \pi \int_o^{\pi} \frac{\sin x\ dx}{1+\cos^2 x} - I$

$2I = \pi \int_o^{\pi} \frac{\sin x\ dx}{1+\cos^2 x}$

Use the substitution, $A = \cos x, \frac{dA}{dx} = -\sin x$ at $x = 0, A = 1$ and $at\ x = \pi, A = -1.$

Therefore, $2I = \pi \int_1^{-1} \frac{-dA}{1+A^2}$

Teachers Comment: Majority of the students arrived at this step.

$= \pi \int_{-1}^{1} \frac{dA}{1+A^2}$

$= 2\pi \int_0^1 \frac{dA}{1+A^2}$ as $\frac{1}{1+A^2}$ is an even function.

$= 2\pi [\tan^{-1} A]_0^1$

$= 2\pi (\frac{\pi}{4})$

$$2I = \frac{\pi^2}{2} \; gives \; I = \frac{\pi^2}{4}$$

Teachers Comment:

- Only some students arrived at this step successfully.
- Several algebraic errors were made in this question.

(b) If $\underset{\sim}{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\underset{\sim}{b} = -m\mathbf{i} + \mathbf{j} + 2\boldsymbol{k}$, where $m$ is a real constant. Calculate the value(s) of $m$ so that the vector $\underset{\sim}{a} - \underset{\sim}{b}$ is perpendicular to $\underset{\sim}{b}$. **2**

$$\underset{\sim}{a} - \underset{\sim}{b} = \begin{pmatrix} 2 - -m \\ -1 - 1 \\ 3 - 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 + m \\ -2 \\ 1 \end{pmatrix}$$

As $\underset{\sim}{a} - \underset{\sim}{b} \perp \underset{\sim}{b}$. Therefore, $(\underset{\sim}{a} - \underset{\sim}{b}).\underset{\sim}{b} = 0$

Which gives $(m + 2) \times -m + 1 \times -2 + 1 \times 2 = 0$

$-m(m + 2) = 0$ gives $m = 0$ or $m = -2$.

Teachers Comment:

- Many students got the correct answers with correct working.
- However, very few calculated $\underset{\sim}{a} - \underset{\sim}{b}$ incorrectly and as a result, got different answers for $m$.

(c)

i. Prove that $\sqrt{3}$ is an irrational number. **2**

Let us assume $\sqrt{3}$ is a rational number. Then $\sqrt{3} = \frac{p}{q}$, where $HCF(p, q) = 1$

And both $p, q$ are integers.

Squaring both sides, $\frac{p^2}{q^2} = 3$ this gives $p^2 = 3q^2$ , $p^2$ is a multiple of 3 and is an integer this implies, $p = 3k$, $k$ any integer. $(3k)^2 = 3q^2$gives, $q^2 = 3k^2$. Hence $q$ is also a multiple of 3. Which means $HCF(p, q) = 3$. This contradicts the assumption that $HCF(p, q) = 1$.

Therefore, $\sqrt{3}$ is an irrational number.

Teachers Comment:

3

- Majority of students did well in this question.
- Few students meant that 3 is the common factor or $HCF(p,q) = 3$, but did not write in the paper, as a result, lost a mark.

ii. Consider the recurrence relation $T_1 = 2, T_n = \sqrt{1 + T_{n-1}}$ for $n \geq 2$.
Using the mathematical induction, prove that $T_n$ is an irrational number for all $n \geq 2$.

For $n = 2, T_2 = \sqrt{1 + T_1}$
$= \sqrt{3}$

Which is an irrational number using part (i).

Teachers Comment:

- Majority of students did well in this question.

Let the result is true for $n = k.\ k \geq 2.\ T_k = \sqrt{1 + T_{k-1}}$ is an irrational number.

To prove that the result is true for $n = k + 1$. That is, $T_{k+1} = \sqrt{1 + T_k}$ is an irrational number.

Let us assume that $T_{k+1} = \sqrt{1 + T_k}$ is a rational number.

$\sqrt{1 + T_k} = \frac{p}{q}$, $HCF(p,q) = 1$ $p, q$ both integers.

Squaring both sides, $1 + T_k = \frac{p^2}{q^2}$, this means $T_k = \frac{p^2}{q^2} - 1$
Since $\frac{p^2}{q^2} - 1 = \frac{p^2 - q^2}{q^2}$, which is rational as $p, q$ both rational. As a result, $T_k$ is a rational number which contradicts the assumption.
Therefore,

$T_n = \sqrt{1 + T_{n-1}}$ for $n \geq 2$
Is an irrational number.

Teachers Comment:

- Majority of students did not do well in this question.

(d) A car is initially at rest at a point $A$ which is 1 km to the right of Woolworths in Seven Hills Plaza. The car then starts moving in a straight line towards Woolworths.

For $x \neq 0$, the acceleration of the particle is given by $-\frac{k}{x^2}$, where $x$ is the distance (in kilometres) from the shop and $k$ is a positive constant.

i. Prove that $\frac{dx}{dt} = -\sqrt{\frac{2k(1-x)}{x}}$. **2**

$\ddot{x} = -\frac{k}{x^2}$ given. Also, $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{k}{x^2}$

Integrating with respect to $x$,

$\frac{1}{2}v^2 = \frac{k}{x} + C$ where $C$ is a constant of integration.

At $t = 0, x = 1, v = 0$, gives $C = -k$.

Therefore, $\frac{1}{2}v^2 = \frac{k}{x} - k$

$v^2 = 2k\left(\frac{1}{x} - 1\right)$ or $v^2 = 2k\left(\frac{1-x}{x}\right)$

Therefore, $v = \pm\sqrt{\frac{2k(1-x)}{x}}$. Since the particle is moving to the left,

therefore $v = -\sqrt{\frac{2k(1-x)}{x}}$, as $v = \frac{dx}{dt}$

$$\frac{dx}{dt} = -\sqrt{\frac{2k(1-x)}{x}}$$

Teachers Comment:

Well done!! **2**

ii. Use the substitution $x = \cos^2\theta$, show that the time taken to reach a distance $D$ kilometres from Woolworths is given by

$$t = \sqrt{\frac{2}{k}}\int_0^{\cos^{-1}\sqrt{D}} \cos^2\theta \; d\theta$$

From part (i)

$$\frac{dx}{dt} = -\sqrt{\frac{2k(1-x)}{x}}$$

Rearranging

$\frac{dx}{\sqrt{\frac{1-x}{x}}} = -\sqrt{2k}dt$ variable separable form

Let $x = \cos^2\theta . \frac{dx}{d\theta} = 2\cos\theta \times -\sin\theta$.

Also, $t = 0, x = 1, \theta = 0$ and at $t, x = D, \theta = \cos^{-1}\sqrt{D}$.

Integrating both sides

$$\int_1^D \frac{dx}{\sqrt{\frac{1-x}{x}}} = \int_0^t -\sqrt{2k}dt$$

$$\int_0^{\cos^{-1}\sqrt{D}} \frac{\cos\theta \times 2\cos\theta \times -\sin\theta \; d\theta}{\sqrt{1-\cos^2\theta}} = \int_0^t -\sqrt{2k}dt \quad , \sin^2\theta = 1 - \cos^2\theta$$

$$\int_0^{\cos^{-1}\sqrt{D}} \frac{\cos\theta \times 2\cos\theta \times -\sin\theta \; d\theta}{\sin\theta} = -\sqrt{2k}t$$

$$-\int_0^{\cos^{-1}\sqrt{D}} 2\cos^2\theta \; d\theta = -\sqrt{2k}t$$

Cancelling negative signs on both sides and rearranging,

$$t = \sqrt{\frac{2}{k}} \int_0^{\cos^{-1}\sqrt{D}} \cos^2\theta \;\; d\theta$$

Teachers Comment: 2

Well done!!

iii. Show that $t = \sqrt{\frac{1}{2k}}\left(\sqrt{D - D^2} + \cos^{-1}\sqrt{D}\right)$.

Using double angle formula, $\cos 2\theta = 2\cos^2\theta - 1$ , $\cos^2\theta = (\cos 2\theta + 1) \div 2$

Substitute it in the expression for time in part (ii)

$$t = \sqrt{\frac{2}{k}} \int_0^{\cos^{-1}\sqrt{D}} (\cos 2\theta + 1) \div 2 \;\; d\theta$$

$$= \sqrt{\frac{2}{k}} \left(\frac{\sin 2\theta}{4} + \frac{\theta}{4}\right)_0^{\cos^{-1}\sqrt{D}}$$

$= \sqrt{\frac{2}{k}}\left(\sin 2(\cos^{-1}\sqrt{D}\right) \div 4\ + \cos^{-1}\sqrt{D} \div 2) \quad$ Let $A = \cos^{-1}\sqrt{D}$ then $\cos A = \sqrt{D}$

therefore, $\sin A = \sqrt{1-D}$, $\sin 2\theta = 2\sin\theta cos\theta$

Sub in the values in the expression for time

$$t = \sqrt{\frac{2}{k}}\left(2\sqrt{1-D}\right) \times \sqrt{D} \div 4 + \cos^{-1}\sqrt{D}\ \div 2)$$

$$t = \sqrt{\frac{1}{2k}}\left(\sqrt{D-D^2} + \cos^{-1}\theta\right)$$

Hence proved.

Teachers Comment:

- Majority of students did well.
- Some students lost mark(s) due to not showing where the values of $\cos A = \sqrt{D}$ therefore, $\sin A = \sqrt{1-D}$,
- Overall well done!!

**End of Question 15**

**Question 16** (13 marks) Use a separate Writing Booklet for Question 16.

(a) The tide at a harbour can be modelled using simple harmonic motion. At the harbour, high tide is 12 metres and low tide is at 2 metres. It takes 4 hours to go from low tide to high tide. Initially at 2 am, it is at low tide. Let *t* be measured in hours since 2 am.

i. Show that $x = 7 - 5\ \cos\left(\frac{t\pi}{4}\right)$. **3**

$x = a\cos(nt+\alpha) + x_0$ , $x_0 = 7$ (centre of motion =(12+2)/7)

$n = 8,\ T = \frac{2\pi}{n}$ gives $T = \frac{\pi}{4}$. And amplitude ($a$) =5.

Therefore, $x = 7 + 5\cos(nt)$, initial phase $\alpha = 0$.

At $t = 0, x = 2$ , it is only possible if $x = 7 - 5\cos(nt)$,

OR

As the wave starts from extreme and goes from low to high, therefore

$$x = 7 - 5\ \cos\left(\frac{t\pi}{4}\right)$$

3

Teachers Comment:

- Some students could show the amplitude, period and the expression correctly, stating from low tide to high tide. Well done!!

- Overall well done.

ii. A ship needs at least 5 metres of depth of water to safely enter the harbour. Find the earliest time that the ship may enter the harbour.

$5 = 7 - 5\cos(\frac{\pi}{4}t),$

$\cos\frac{\pi}{4}t = \frac{2}{5}$

$\frac{\pi}{4}t = \cos^{-1}\frac{2}{5}$

$t = 1.476$

$t = 1$ hour and 29 minutes. Therefore at 3.29 am, the ship may enter the harbour.

Teachers Comment:

Well done!!

(b) i. Find the real numbers $a, b$ and $c$ such that

$$\frac{5-5x^2}{(1+2x)(1+x^2)} \equiv \frac{a}{1+2x} + \frac{bx+c}{1+x^2}$$ 2

$$5-5x^2 \equiv a(1+x^2) + (bx+c)(1+2x)$$

For $x = -\frac{1}{2}$,

$5-5\left(-\frac{1}{2}\right)^2 \equiv a\left(1+\left(-\frac{1}{2}\right)^2\right) + (b(-\frac{1}{2})+c)(1+2(-\frac{1}{2}))$ which gives $a = 3$.

For $x = 0$,

$5-5(0)^2 \equiv a(1+(0)^2) + (b(0)+c)(1+2(0))$ which gives, $a + c = 5$, hence $c = 2$.

For $x = 1$,

$5 - 5(1)^2 \equiv a(1 + (1)^2) + (b(1) + c)(1 + 2(1))$ which gives $b = -2 - c$, hence $b = -4$.

Teachers Comment:

- Well done!! **3**

ii. Show that $\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)} dx = \frac{1}{2}\left(\pi + \ln\left(\frac{27}{16}\right)\right)$.

Using part (i)

$$\frac{5 - 5x^2}{(1 + 2x)(1 + x^2)} \equiv \frac{3}{1 + 2x} + \frac{-4x + 2}{1 + x^2}$$

Therefore,

$$\begin{aligned}
\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)} dx &= \int_0^1 \frac{3}{1+2x} dx + \int_0^1 \frac{-4x+2}{1+x^2} dx \\
&= [3(\ln(1 + 2x) \div 2]_0^1 - 2[\ln(1 + x^2)]_0^1 + 2[\tan^{-1} x]_0^1 \\
&= \frac{3}{2}(\ln 3 - \ln 1) - 2(\ln 2 - \ln 1) + 2(\tan^{-1} 1 - \tan^{-1} 0) \\
&= \frac{3}{2}\ln 3 - 2\ln 2 + \frac{2\pi}{4} \\
&= \frac{1}{2}(3\ln 3 - 4\ln 2) + \frac{\pi}{2} \\
&= \frac{1}{2}(\ln 27 - \ln 16) + \frac{\pi}{2} \\
&= \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right) \\
&= RHS
\end{aligned}$$

**2**

Teachers Comment:

- Few students did some algebraic mistakes and lost mark(s).
- Overall, well done!!

iii. Hence, evaluate $\int_0^{\frac{\pi}{4}} \frac{\cos 2x \, dx}{1+2 \sin 2x+\cos 2x}$ using the substitution $t = \tan x$.

$t = \tan x$, $\frac{dt}{dx} = \sec^2 x$ therefore, $dx = \frac{dt}{1+t^2}$. At $x = 0, t = 0$ and at $x = \frac{\pi}{4}$, $t = 1$

$$\int_0^{\frac{\pi}{4}} \frac{\cos 2x\, dx}{1+2\ \sin 2x+\cos 2x} = \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1+2\left(\frac{2t}{1+t^2}\right)+\frac{(1-t^2)}{1+t^2}} \times \frac{dt}{1+t^2}$$

$$= \int_0^1 \frac{1-t^2}{(1+t^2)(2+4t)} dt$$

$$= \frac{1}{10}\int_0^1 \frac{5-5t^2}{(1+t^2)(1+2t)} dt$$

$= \frac{1}{20}\left(\pi + \ln\frac{27}{16}\right)$ using part ii), hence Proved.

Teachers Comment:

- Few students did some algebraic mistakes and lost mark(s).
- Overall, well done!!

**End of Question 16**